

SET-VALUED SORTING INDEX AND JOINT EQUIDISTRIBUTIONS

SEN-PENG EU, YUAN-HSUN LO, AND TSAI-LIEN WONG

ABSTRACT. Recently Petersen defined a new Mahonian index sor over the symmetric group \mathfrak{S}_n and proved that $(\text{inv}, \text{rmin})$ and (sor, cyc) have the same joint distribution. Foata and Han proved that the pairs of set-valued statistics $(\text{Cyc}, \text{Rmil})$, $(\text{Cyc}, \text{Lmap})$, $(\text{Rmil}, \text{Lmap})$ have the same joint distribution over \mathfrak{S}_n .

In this paper we introduce the set-valued statistics Inv , Lmil , Sor and Lmic_1 and generalize simultaneously results of Petersen and Foata-Han and find many equidistributed triples of set-valued statistics and quadruples of statistics.

1. INTRODUCTION

Let \mathfrak{S}_n be the symmetric group of $[n] := \{1, 2, \dots, n\}$. For $\sigma = \sigma_1\sigma_2 \dots \sigma_n \in \mathfrak{S}_n$, define the *inversion* statistic inv by

$$\text{inv}(\sigma) := \#\{(i, j) : i < j \text{ and } \sigma_i < \sigma_j\},$$

and the *cycle* statistic $\text{cyc}(\sigma)$ by

$$\text{cyc}(\sigma) := \text{the number of cycles in the cycle decomposition of } \sigma.$$

We say a permutation statistic over \mathfrak{S}_n is *Mahonian* if it is equidistributed with inv , and is *Stirling* if with cyc . For example, it is well known that the *right to left minimum* statistic rmin , defined by

$$\text{rmin}(\sigma) := \#\{\sigma_i : \sigma_i < \sigma_j \text{ for all } j > i\},$$

is Stirling.

Recently Petersen found a new Mahonian statistic sor , called the *sorting index* (see Section 2 for definition), and proved the following:

Theorem 1.1 (Petersen [8]). *The pairs of statistics $(\text{inv}, \text{rmin})$ and (sor, cyc) have the same joint distribution over \mathfrak{S}_n . Also, we have*

$$\begin{aligned} \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} x^{\text{rmin}(\sigma)} &= \sum_{\sigma \in \mathfrak{S}_n} q^{\text{sor}(\sigma)} x^{\text{cyc}(\sigma)} \\ &= x \prod_{r=2}^n (x + [r]_q - 1), \end{aligned}$$

where $[r]_q := 1 + q + q^2 + \dots + q^{r-1}$.

2010 *Mathematics Subject Classification.* 05A05, 05A19.

Key words and phrases. sorting index, Mahonian statistics, Stirling statistics, joint distribution, equidistribution.

Partially supported by National Science Council, Taiwan under grants NSC 101-2115-M-003-013-MY3 (S.-P. Eu), 102-2811-M-003-025 (Y.-H. Lo) and NSC 102-2115-M-110-006-MY2 (T.-L. Wong).

A combinatorial proof is found by Chen et al. [2] via a bijection $\phi : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ such that

$$(\text{inv}, \text{rmin})\sigma = (\text{sor}, \text{cyc})\phi(\sigma),$$

where $(\text{inv}, \text{rmin})\sigma$ means $(\text{inv}(\sigma), \text{rmin}(\sigma))$. The bijection ϕ turns out to be of the form

$$\phi = B(\sigma)^{-1} \circ A(\sigma),$$

a composition of B -code and A -code introduced by Foata and Han [6]. By defining the set-values statistics *right-to-left minimum letters* Rmil , *left-to-right maximum places* Lmap and the *cycle set* Cyc respectively by

$$\text{Rmil}(\sigma) := \{\sigma_i : \sigma_i < \sigma_j \text{ for all } j > i\},$$

$$\text{Lmap}(\sigma) := \{i : \sigma_i > \sigma_j \text{ for all } i > j\},$$

and

$$\text{Cyc}(\sigma) := \{\text{the smallest number in each cycle of the cycle decomposition}\},$$

Foata and Han derived the following set-valued joint equidistribution results.

Theorem 1.2 (Foata, Han [6]). *The followings hold.*

(1) *For $\sigma \in \mathfrak{S}_n$, we have*

$$(\text{Rmil}, \text{Lmap})\sigma = (\text{Cyc}, \text{Lmap})\phi(\sigma).$$

(2) *The set-valued statistics $(\text{Cyc}, \text{Rmil})$, $(\text{Cyc}, \text{Lmap})$, $(\text{Rmil}, \text{Lmap})$ are symmetric and joint equidistributed over \mathfrak{S}_n .*

The motivation of this work is to generalize above two theorems, to see if there is a set-valued version of Petersen's result or other pairs of set-valued statistics having the same distribution à la Foata and Han. It turns out that we can have them both.

Throughout the paper a statistic is set-valued if and only if the first letter is in capital. By introducing new set-valued statistics Inv , Lmil , Sor and Lmic_1 (see Section 2 for definitions) and corresponding ordinary statistics lmin , lmic_1 , our first main theorem extends simultaneously both Petersen and Foata-Han's results.

Theorem 1.3. *We have:*

(1) *For $\sigma \in \mathfrak{S}_n$, the following holds:*

$$(\text{Inv}, \text{Rmil}, \text{Lmap}, \text{Lmil})\sigma = (\text{Sor}, \text{Cyc}, \text{Lmap}, \text{Lmic}_1)\phi(\sigma).$$

(2) *The quadruple statistics $(\text{inv}, \text{rmin}, \text{lmax}, \text{lmin})$ and $(\text{sor}, \text{cyc}, \text{lmax}, \text{lmic}_1)$ have the same joint distribution over \mathfrak{S}_n , and*

$$\begin{aligned} \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} x^{\text{rmin}(\sigma)} y^{\text{lmin}(\sigma)} &= \sum_{\sigma \in \mathfrak{S}_n} q^{\text{sor}(\sigma)} x^{\text{cyc}(\sigma)} y^{\text{lmic}_1(\sigma)} \\ &= xy \prod_{r=2}^n (x + [r]_q + yq^{r-1} - 1 - q^{r-1}). \end{aligned}$$

Theorem 1.3 generalizes Theorem 1.1 and Theorem 1.2 (1). We will see that Theorem 1.2 (2) will be generalized in a later theorem.

By abuse of terminology, a set-valued statistic is called Mahonian (or Stirling) if the corresponding ordinary statistic is so. Our second result is to find triples of set-valued Stirling statistics having the same joint distribution as $(\text{Rmil}, \text{Lmap}, \text{Lmil})$ and $(\text{Cyc}, \text{Lmic}_1, \text{Lmap})$. By switching between certain set-valued statistics by applying invese, reverse, or complement operations on permutations, we obtain 8 more (and 4 partial) triples of set-valued statistics having the same joint distribution. See Theorem 4.2 for the complete list. From these we obtain two sets of symmetric and equidistributed pairs of set-valued statistics, the first of which includes those three pairs in Theorem 1.2 (2).

The third part of the work is to derive quadruples of statistics which are joint equidistributed with $(\text{inv}, \text{rmin}, \text{lmax}, \text{lmin})$ and $(\text{sor}, \text{cyc}, \text{lmic}_1, \text{lmax})$. Note that the first statistic is Mahonian and the others are Stirling. Again by switching among statistics, in Theorem 5.2 we derive 10 more (and 11 partial) quadruples of statistics having the same joint distribution.

The rest of the paper is organized as follows. Definitions and preliminary results will be put in Section 2. In Section 3 we prove Theorem 1.3. Section 4 is devoted to triples of set-valued statistics, and Section 5 to quadruples of ordinary statistics.

2. PRELIMINARY RESULTS

2.1. *A-code and B-code.* We first introduce the *A-* and *B-code* of Foata and Han [6], which are the key tools of this paper. Given $\sigma \in \mathfrak{S}_n$, define its *Lehmer code* [7] by

$$\text{Leh}(\sigma) := (\ell_1, \ell_2, \dots, \ell_n),$$

where $\ell_i = |\{j : 1 \leq j \leq i, \sigma_j \leq \sigma_i\}|$. Let $\mathcal{L}_n := \{(\ell_1, \ell_2, \dots, \ell_n) : 1 \leq \ell_i \leq i \text{ for } 1 \leq i \leq n\}$. It is clear that $\text{Leh} : \mathfrak{S}_n \rightarrow \mathcal{L}_n$ is a bijection. The *A-code* of a permutation σ is defined by

$$A(\sigma) := \text{Leh}(\sigma^{-1}).$$

For example, let $\sigma = 2413765$. Then $\sigma^{-1} = 3142765$ and $A(\sigma) = (1, 1, 3, 2, 5, 5, 5)$.

The *B-code* of σ is defined in the following way. For each $i = 1, \dots, n$, let $k_i \geq 1$ be the smallest integer such that $\sigma^{-k_i}(i) \leq i$. Define

$$B(\sigma) := (b_1, b_2, \dots, b_n) \text{ with } b_i = \sigma^{-k_i}(i).$$

Equivalently, $B(\sigma)$ can be determined from the cycle decomposition of σ . Assume that i appears in a cycle c . If i is the smallest element of c , then set $b_i = i$; otherwise, choose b_i to be the first element j in c with respect to the reverse direction such that $j < i$. For example, let $\sigma = 2431756 = (124)(3)(576)$. Then $B(\sigma) = (1, 1, 3, 2, 5, 5, 5)$. By definition it is easy to see that *B-code* is a bijection from \mathfrak{S}_n to \mathcal{L}_n .

2.2. The set-valued statistic Lmic_1 . Define the *left-to-right minimum letters* statistic Lmil , the set-valued *left-to-right minimum places* statistic Lmip and the *left to right minimum* statistic lmin respectively by

$$\text{Lmil}(\sigma) := \{\sigma_i : \sigma_i < \sigma_j \text{ for all } j < i\},$$

$$\text{Lmip}(\sigma) := \{i : \sigma_i < \sigma_j \text{ for all } j < i\},$$

and

$$\text{lmin}(\sigma) := \#\text{Lmil}(\sigma) \quad (\text{or } \#\text{Lmip}(\sigma)).$$

It is easy to see that $\text{Lmil}(\sigma) = \text{Lmip}(\sigma^{-1})$.

For an integer sequence $(\ell_1, \dots, \ell_n) \in \mathcal{L}_n$, define $O((\ell_1, \dots, \ell_n)) := \{i : \ell_i = 1\}$, the set of indices with values 1. It turns out that Lmip and Lmil correspond to Lehmer code and A -code respectively. The proof of the following lemma is directly by definition.

Lemma 2.1. *We have*

$$\text{Lmip}(\sigma) = O(\text{Leh}(\sigma)) \quad \text{and} \quad \text{Lmil}(\sigma) = O(A(\sigma)). \quad (1)$$

Hence it is natural to consider the statistic corresponding to B -code. Define the set-valued statistic Lmic_1 by

$$\text{Lmic}_1(\sigma) := O(B(\sigma)). \quad (2)$$

For example, $\text{Lmic}_1(579328164) = O((1, 1, 3, 3, 1, 6, 2, 6, 3)) = \{1, 2, 5\}$.

We have the following combinatorial interpretation of Lmic_1 , which explains the somewhat awkward notation, standing for the *left-to-right minimum of the shifted cycle containing 1*.

Lemma 2.2. *For $\sigma \in \mathfrak{S}_n$, write the cycle containing 1 in the way that 1 is at the end of the cycle and denote the resulting cycle \vec{c} . Then*

$$\text{Lmic}_1(\sigma) = \text{Lmil}(\vec{c})$$

by regarding \vec{c} as a word.

Proof. By the definition of B -code, $b_i = 1$ if and only if $i \in \vec{c}$ and all letters on the left of i in \vec{c} are larger than i . In other words, i is a left-to-right minimum letter in \vec{c} . Then we have $O(B(\sigma)) = \text{Lmil}(\vec{c})$. \square

For the running example, $\sigma = 579328164 = (1527)(394)(68)$ and the shifted cycle is $\vec{c} = (5271)$, hence $\text{Lmic}_1(579328164) = \text{Lmil}(5271) = \{1, 2, 5\}$.

Also, Lmil and Lmic_1 are related via ϕ .

Lemma 2.3. *We have*

$$\text{Lmil}(\sigma) = \text{Lmic}_1(\phi(\sigma)).$$

Proof. From (1), (2) and the definition of ϕ , we have

$$\text{Lmic}_1(\phi(\sigma)) = O(B(\phi(\sigma))) = O(A(\sigma)) = \text{Lmil}(\sigma).$$

\square

2.3. The set-valued statistics Inv and Sor . The goal of this subsection is to define and investigate the set-valued statistics Sor and Inv . First we need the concept of the *induced set*.

Definition 2.4. *Given $(\ell_1, \ell_2, \dots, \ell_n) \in \mathcal{L}_n$, define its induced set $\langle(\ell_1, \ell_2, \dots, \ell_n)\rangle$ according to the following algorithm:*

- (1) Set S, U with the initial values $S = \{1, 2, \dots, n\}$ and $U = \emptyset$.

- (2) For i from n down to 1 do the followings:
- (a) let ℓ'_i be the ℓ_i -th smallest element among S ,
 - (b) add ordered pairs (ℓ'_i, j) into U for those $j \in S$ with $j > \ell'_i$. If there is no such j then skip this step.
 - (c) delete ℓ'_i from S .
- (3) Define $\langle(\ell_1, \ell_2, \dots, \ell_n)\rangle := U$.

For example, for $(1, 1, 3, 2, 5, 5, 5) \in \mathcal{L}_7$ we have

| i | ℓ'_i | S | U |
|-----|-----------|---------------------|--|
| | | 1, 2, 3, 4, 5, 6, 7 | \emptyset |
| 7 | 5 | 1, 2, 3, 4, *, 6, 7 | (5,6), (5,7) |
| 6 | 6 | 1, 2, 3, 4, *, *, 7 | (5,6), (5,7), (6,7) |
| 5 | 7 | 1, 2, 3, 4, *, *, * | (5,6), (5,7), (6,7) |
| 4 | 2 | 1, *, 3, 4, *, *, * | (5,6), (5,7), (6,7), (2,3), (2,4) |
| 3 | 4 | 1, *, 3, *, *, *, * | (5,6), (5,7), (6,7), (2,3), (2,4) |
| 2 | 1 | *, *, 3, *, *, *, * | (5,6), (5,7), (6,7), (2,3), (2,4), (1,3) |
| 1 | 3 | *, *, *, *, *, *, * | (5,6), (5,7), (6,7), (2,3), (2,4), (1,3) |

Thus $\langle(1, 1, 3, 2, 5, 5, 5)\rangle = \{(5, 6), (5, 7), (6, 7), (2, 3), (2, 4), (1, 3)\}$.

Now we review the *sorting index* sor of Petersen [8]. Given σ , decompose it uniquely into the product of transpositions $\sigma = (i_1 j_1)(i_2 j_2) \dots (i_k j_k)$ with $j_1 < j_2 < \dots < j_k$ and $i_r < j_r$ for $1 \leq r \leq k$, and then define

$$\text{sor}(\sigma) := \sum_{r=1}^k (j_r - i_r).$$

For example, since $\sigma = 2431765 = (12)(24)(56)(57)$ we have $\text{sor}(\sigma) = (2-1) + (4-2) + (6-5) + (7-5) = 6$. In other words, $\text{sor}(\sigma)$ measures the total distance of the letters needed to move during the bubble-sorting process. In this example, we have

$$2431756 \xrightarrow{(57)} 2431657 \xrightarrow{(67)} 2431567 \xrightarrow{(24)} 2134567 \xrightarrow{(12)} 1234567.$$

In [2] it is proved that $\text{inv}(\sigma) = \text{sor}(\phi(\sigma))$ and

$$\text{sor}(\sigma) = \sum_{i=1}^n (i - b_i), \quad (3)$$

which clarifies the relation between sorting index and the B -code (b_1, b_2, \dots, b_n) of σ . Observe that in the step (2)(b) of 2.4 we add exactly $(i - \ell_i)$ ordered pairs into U for each i , hence from (3) it makes sense to define the set-valued statistic *sorting set* Sor by

$$\text{Sor}(\sigma) := \langle B(\sigma) \rangle. \quad (4)$$

As for the Inv , since $\text{inv}(\sigma) := \#\{(i, j) : i < j \text{ and } \sigma_i < \sigma_j\}$, it is natural to define the set-valued statistic *inversion set* Inv by

$$\text{Inv}(\sigma) := \{(i, j) : i < j \text{ and } \sigma_i < \sigma_j\}.$$

Similar to the relation between Sor and B -code, we have the following:

Proposition 2.5. *For $\sigma \in \mathfrak{S}_n$, we have*

$$\text{Inv}(\sigma) = \langle A(\sigma) \rangle. \quad (5)$$

Proof. Let $\sigma = \sigma_1 \cdots \sigma_n$ be the permutation with $A(\sigma) = (a_1, \dots, a_n)$. Since $A(\sigma) = \text{Leh}(\sigma^{-1})$, hence $\sigma_n^{-1} = a_n$, $\sigma_{n-1}^{-1} = (a_{n-1})$ -th smallest element in $[n] \setminus \{\sigma_n^{-1}\}$. In general, for $1 < i < n$ we have

$$\sigma_{n-i}^{-1} = (a_{n-i})\text{-th smallest element in } [n] \setminus \{\sigma_n^{-1}, \dots, \sigma_{n-i+1}^{-1}\}.$$

Thus σ can be rebuilt from $A(\sigma)$ as follows. At the initial stage there are n vacancies from left to right. For i from n down to 1, we recursively put letter i into the a_i -th vacancy from the left. The resulting permutation is exactly σ .

For example, if $A(\sigma) = (1, 1, 3, 2, 5, 5, 5)$, then σ can be recovered in the following way:

$$\text{-----} \rightarrow \text{----}7\text{--} \rightarrow \text{----}76\text{--} \rightarrow \text{----}765 \rightarrow \text{--}4\text{--}765 \rightarrow \text{--}43765 \rightarrow 2\text{--}43765 \rightarrow 2143765$$

Now observe that in each step above, the position we choose for the letter i is exactly ℓ'_i in Definition 2.4. In other words, we can synchronize the rebuilding of σ and the construction of the induced set $\langle A(\sigma) \rangle$. Moreover, note that the ordered pair (ℓ'_i, j) is added to U if and only if it is an inversion of σ , for the letter i must be larger than σ_j . Thus it must have $\text{Inv}(\sigma) = \langle A(\sigma) \rangle$. \square

And finally there is a set version of $\text{inv}(\sigma) = \text{sor}(\phi(\sigma))$:

Lemma 2.6. *We have*

$$\text{Inv}(\sigma) = \text{Sor}(\phi(\sigma)).$$

Proof. From (4), (5) and the definition of ϕ , we have

$$\text{Sor}(\phi(\sigma)) = \langle B(\phi(\sigma)) \rangle = \langle A(\sigma) \rangle = \text{Inv}(\sigma).$$

\square

3. PROOF OF THEOREM 1.3

Proof of Theorem 1.3. (1) is obtained by combining Lemma 2.3, 2.6 and Theorem 1.2. The first statement of (2) is directly from (1) and in the following we look at the generating function. For $n \geq 1$ let

$$F_n(q, x, y) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{sor}(\sigma)} x^{\text{cyc}(\sigma)} y^{\text{lmic}_1(\sigma)}.$$

It is clear that $F_1(q, x, y) = xy$. We claim that for $n \geq 2$ one has

$$F_n(q, x, y) = xy \prod_{r=2}^n (x + [r]_q + yq^{r-1} - 1 - q^{r-1}).$$

Let t_{ij} denote the transposition (ij) and let $\eta_1 = 1, \eta_2 = 1 + t_{12}$ and

$$\eta_j = 1 + \sum_{i < j} t_{ij}$$

for $j \geq 3$. Petersen [8] showed that

$$\eta_1 \eta_2 \cdots \eta_n = \sum_{\sigma \in \mathfrak{S}_n} \sigma. \quad (6)$$

Now define the linear map $\Theta : \mathfrak{S}_n \rightarrow \mathbb{Z}[q, x, y]$ by $\Theta(\sigma) := q^{\text{sor}(\sigma)} x^{\text{cyc}(\sigma)} y^{\text{lmic}_1(\sigma)}$. Hence by (6) it suffices to show that

$$\Theta(\eta_1 \eta_2 \cdots \eta_n) = xy \prod_{r=2}^n (x + [r]_q + yq^{r-1} - 1 - q^{r-1}).$$

It is easy to see that $\Theta(\eta_1) = xy$ and $\Theta(\eta_1 \eta_2) = xy(x + yq)$. Let $n \geq 3$. We proceed by induction. Suppose

$$\Theta(\eta_1 \eta_2 \cdots \eta_{n-1}) = xy \prod_{r=2}^{n-1} (x + [r]_q + yq^{r-1} - 1 - q^{r-1}).$$

Take $\sigma = \sigma_1 \sigma_2 \cdots \sigma_{n-1} \in \mathfrak{S}_{n-1}$. It can be embedded in \mathfrak{S}_n as $\sigma = \sigma_1 \sigma_2 \cdots \sigma_{n-1} n$. Let $\sigma' := \sigma t_{i n}$ for some $1 \leq i \leq n$ and it is clear that $\text{sor}(\sigma') = \text{sor}(\sigma) + (n - i)$.

Assume that the cycle decomposition of σ is $c_1 \cdots c_m$ for some m and c_t contains the letter i . Hence $\sigma' = c_1 \cdots c_m(n)$ if $i = n$ and $c_1 \cdots c'_t \cdots c_m$ if $i \neq n$, where $c'_t = (\dots, i, n, \sigma(i), \dots)$. Thus

$$\text{cyc}(\sigma') = \begin{cases} \text{cyc}(\sigma) + 1 & \text{if } i = n, \\ \text{cyc}(\sigma) & \text{otherwise;} \end{cases}$$

and

$$\text{lmic}_1(\sigma') = \begin{cases} \text{lmic}_1(\sigma) + 1 & \text{if } i = 1, \\ \text{lmic}_1(\sigma) & \text{otherwise.} \end{cases}$$

So we have

$$\begin{aligned} \Theta(\sigma \cdot \eta_n) &= \Theta(\sigma t_{n n} + \sigma t_{n-1 n} + \cdots + \sigma t_{1 n}) \\ &= \Theta(\sigma)(x + q + q^2 + \cdots + q^{n-2} + yq^{n-1}), \end{aligned}$$

and therefore

$$\begin{aligned} \Theta(\eta_1 \eta_2 \cdots \eta_n) &= \Theta \left(\sum_{\sigma \in \mathfrak{S}_n, \sigma(n)=n} \sigma \cdot \eta_n \right) \\ &= \sum_{\sigma \in \mathfrak{S}_n, \sigma(n)=n} \Theta(\sigma \cdot \eta_n) \\ &= (x + q + q^2 + \cdots + q^{n-2} + yq^{n-1}) \sum_{\sigma \in \mathfrak{S}_{n-1}} \Theta(\sigma) \\ &= (x + [n]_q + yq^{n-1} - 1 - q^{n-1}) \Theta(\eta_1 \eta_2 \cdots \eta_{n-1}). \end{aligned}$$

The proof is then completed by induction. \square

4. SET-VALUED JOINT EQUIDISTRIBUTION

In this section we seek for more set-valued statistics having the same joint distribution as $(\text{Rmil}, \text{Lmil}, \text{Lmap})$ and $(\text{Cyc}, \text{Lmic}_1, \text{Lmap})$. In the introduction we have defined Rmil (right-to-left minimum letters) and Lmap (left-to-right maximum places) while in Section 2.2 Lmil (left-to-right minimum letters) and Lmip (left-to-right minimum places). Similarly we can defined set-valued statistics Rmip , Lmal , Rmap and Rmal by

$$\text{Rmip}(\sigma) := \{i : \sigma_i < \sigma_j \text{ for all } j > i\},$$

$$\text{Lmal}(\sigma) := \{\sigma_i : \sigma_i > \sigma_j \text{ for all } j < i\},$$

$$\text{Rmap}(\sigma) := \{i : i > \sigma_j \text{ for all } j > i\},$$

$$\text{Rmal}(\sigma) := \{i : \sigma_i > \sigma_j \text{ for all } j > i\}.$$

The idea is quite simple: we look at the relations between these statistics by performing operations of “inverse”, “complement”, or “reverse” on permutations.

For $\sigma = \sigma_1 \sigma_2 \dots \sigma_n \in \mathfrak{S}_n$, let σ^{-1} denote its *inverse*,

$$\sigma^r := (\sigma_n, \sigma_{n-1}, \dots, \sigma_1)$$

its *reverse* and

$$\sigma^c := (n+1-\sigma_1, n+1-\sigma_2, \dots, n+1-\sigma_n)$$

its *complement*. For example, if $\sigma = 364152$, then $\sigma^{-1} = 461352$, $\sigma^r = 251463$ and $\sigma^c = 413625$. It is clear that the mappings $i, r, c : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$, defined by $i(\sigma) := \sigma^{-1}$, $r(\sigma) := \sigma^r$ and $c(\sigma) := \sigma^c$, are bijections.

Given two set-valued statistics Stat_1 , Stat_2 and one bijection $\chi : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$, we say

$$\text{Stat}_1 \xrightarrow{\chi} \text{Stat}_2$$

if $\text{Stat}_1(\sigma) = \text{Stat}_2(\chi(\sigma))$ for all $\sigma \in \mathfrak{S}_n$. Also we define $\text{Stat}^* := \{n+1-i : i \in \text{Stat}\}$ for a set-valued statistic Stat , if applicable.

It turns out that these eight set-valued statistics are related via the mappings i, r and c . The proof of the following proposition is straightforward and is omitted.

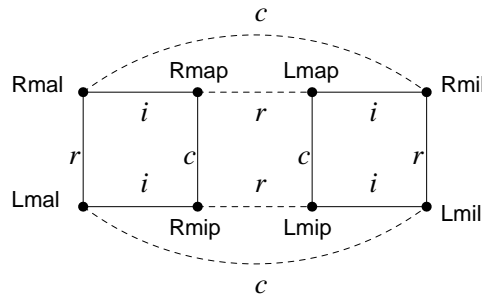


FIGURE 1. Relations between eight statistics

Proposition 4.1. *The relations between the eight set-valued statistics Lmil , Lmip , Rmil , Rmip , Lmal , Lmap , Rmal and Rmap are illustrated by the graph (with edges solid or dotted and labeled by i, r or c) in Figure 1.*

- (1) A solid edge between Stat_1 and Stat_2 means $\text{Stat}_1(\sigma) = \text{Stat}_2(\chi(\sigma))$ and $\text{Stat}_2(\sigma) = \text{Stat}_1(\chi(\sigma))$ for $\sigma \in \mathfrak{S}_n$, with $\chi = i, r$ or c as labeled.
- (2) A dotted edge between Stat_1 and Stat_2 means $\text{Stat}_1(\sigma) = \text{Stat}_2^*(\chi(\sigma))$ and $\text{Stat}_2(\sigma) = \text{Stat}_1^*(\chi(\sigma))$ for $\sigma \in \mathfrak{S}_n$, with $\chi = i, r$ or c as labeled.

For examples, we have $\text{Rmap} \xrightarrow{i} \text{Rmal}$, $\text{Rmap} \xrightarrow{r} \text{Lmap}^*$, and $\text{Rmap} \xrightarrow{c} \text{Rmip}$.

We come to the main result of this section.

Theorem 4.2. *The following triples of set-valued statistics have the same joint distribution over \mathfrak{S}_n . A dash means the statistic is omitted.*

- | | | |
|---------------------------|-------------------------------------|-------------------------|
| (1) (Rmil, Lmil, Lmap) | (2) (Cyc, Lmic ₁ , Lmap) | (3) (Lmap, Lmip, Rmil) |
| (4) (Rmip*, Rmap*, Lmal*) | (5) (Lmal*, Rmal*, Rmip*) | (6) (Lmil, Rmil, Rmap*) |
| (7) (Rmal*, Lmal*, Lmip) | (8) (Rmap*, Rmip*, Lmil) | (9) (Lmip, Lmap, Rmal*) |
| (10) (Lmap, -, Cyc) | (11) (Lmic ₁ , Cyc, -) | (12) (Cyc, -, Rmil) |
| (13) (Rmil, -, Cyc), | | |

Proof. By Proposition 4.1, we have

$$\begin{aligned} (1) &\xrightarrow{i} (3), & (1) &\xrightarrow{i \circ r \circ c} (4), & (1) &\xrightarrow{r \circ c} (5), & (1) &\xrightarrow{r} (6), \\ (1) &\xrightarrow{c} (7), & (1) &\xrightarrow{i \circ r} (8), & (1) &\xrightarrow{i \circ c} (9). \end{aligned}$$

Therefore, (3) to (9) are joint equidistributed with (1). Moreover, by Theorem 1.3, we have

$$(1) \xrightarrow{\phi} (2), \quad (3) \xrightarrow{\phi} (10), \quad (6) \xrightarrow{\phi} (11).$$

Finally, (2) \xrightarrow{i} (12) and (10) \xrightarrow{i} (13) follow from the fact that $\text{Cyc}(\sigma) = \text{Cyc}(\sigma^{-1})$, and the proof is completed. \square

From the theorem we can read off the following many pairs of set-valued statistics which are symmetric and joint equidistributed. Note that (1) includes those pairs of Foata and Han.

Corollary 4.3. *In each of the following items, the pairs of set-valued statistics are symmetric and joint equidistributed over \mathfrak{S}_n .*

- (1) (Rmil, Lmap), (Rmip^d, Lmal^d), (Rmal^d, Lmip), (Rmap^d, Lmil), (Cyc, Rmil) and (Cyc, Lmap)
- (2) (Rmil, Lmil), (Lmap, Lmip), (Rmip^d, Rmap^d), (Rmal^d, Lmal^d), and (Cyc, Lmic₁).

5. JOINT EQUIDISTRIBUTED QUADRUPLES

In this section we look at the ordinary number-valued statistics. The goal is to find quadruples of statistics joint equidistributed with (inv, rmin, lmax, lmin) and (sor, cyc, lmax, lmic₁). Most of the materials in this section are well known. Our contribution is to relate them with the (sor, cyc, lmax, lmic₁) and derive the generating functions with respect to the first three statistics.

The statistics rmin, rmax, lmin, lmax are defined in the obvious way. We may take more familiar Mahonian statistics into consideration. Let

$$\text{Des}(\sigma) := \{i : \sigma_i > \sigma_{i+1}\}$$

be the *descent set* of σ . It is well known that the statistics *major* maj , *inverse major* imaj , *reverse major* rmaj , *charge* chg , and *cocharge* cochg , defined by

$$\begin{aligned}\text{maj}(\sigma) &:= \sum_{i \in \text{Des}} i, \\ \text{imaj}(\sigma) &:= \text{maj}(\sigma^{-1}), \quad \text{rmaj}(\sigma) := \text{maj}(\sigma^{r \circ c}), \\ \text{chg}(\sigma) &:= \sum_{i \in \text{Des}(\sigma^{-1})} (n - i), \quad \text{cochg}(\sigma) := \sum_{i \notin \text{Des}(\sigma^{-1})} (n - i),\end{aligned}$$

are all Mahonian [9].

Again, given two statistics $\text{stat}_1, \text{stat}_2$ and one bijection $\chi : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$, the notation

$$\text{stat}_1 \xrightarrow{\chi} \text{stat}_2$$

means $\text{stat}_1(\sigma) = \text{stat}_2(\chi(\sigma))$ for all $\sigma \in \mathfrak{S}_n$. The equidistribution of maj and inv can be proved combinatorially from the celebrated *fundamental bijection* $\psi : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ of Foata [4], that is, one has

$$\text{maj} \xrightarrow{\psi} \text{inv}.$$

Moreover, a close look of ψ will also show that

$$(\text{rmax}, \text{rmin}) \xrightarrow{\psi} (\text{rmax}, \text{rmin}).$$

We now perform the “inverse-reverse-complement” trick on these statistics.

Lemma 5.1. *We have*

$$(1) \text{maj} \xrightarrow{i} \text{imaj}, \quad (2) \text{maj} \xrightarrow{r \circ c} \text{rmaj}, \quad (3) \text{rmaj} \xrightarrow{i} \text{chg}, \quad (4) \text{chg} \xrightarrow{r} \text{cochg}.$$

Proof. (1), (2) and (4) are obvious. For (3), if $i \in \text{Des}(\sigma^{r \circ c})$, then $(n+1) - \sigma_i^r < (n+1) - \sigma_{i+1}^r$ and thus $\sigma_{n-i} > \sigma_{n-i+1}$, which implies that $n - i \in \text{Des}(\sigma)$. Hence we have

$$\text{rmaj}(\sigma) = \text{maj}(\sigma^{r \circ c}) = \sum_{i \in \text{Des}(\sigma^{r \circ c})} i = \sum_{i \in \text{Des}(\sigma)} (n - i) = \text{chg}(\sigma^{-1}),$$

as desired. □

We are ready to state the main result of this section.

Theorem 5.2. *The following quadruple statistics are joint equidistributed over \mathfrak{S}_n :*

- | | |
|--|--|
| (1) $(\text{inv}, \text{rmin}, \text{lmin}, \text{lmax})$ | (2) $(\text{sor}, \text{cyc}, \text{lmic}_1, \text{lmax})$ |
| (3) $(\text{inv}, \text{lmax}, \text{lmin}, \text{rmin})$ | (4) $(\text{inv}, \text{rmin}, \text{rmax}, \text{lmax})$ |
| (5) $(\text{inv}, \text{lmax}, \text{rmax}, \text{rmin})$ | (6) $(\binom{n}{2} - \text{inv}, \text{lmin}, \text{rmin}, \text{rmax})$ |
| (7) $(\binom{n}{2} - \text{inv}, \text{rmax}, \text{lmax}, \text{lmin})$ | (8) $(\binom{n}{2} - \text{inv}, \text{rmax}, \text{rmin}, \text{lmin})$ |
| (9) $(\binom{n}{2} - \text{inv}, \text{lmin}, \text{lmax}, \text{rmax})$ | (10) $(\text{sor}, \text{lmax}, \text{lmic}_1, \text{cyc})$ |
| (11) $(\binom{n}{2} - \text{sor}, \text{lmic}_1, \text{cyc}, -)$ | (12) $(\binom{n}{2} - \text{sor}, -, \text{lmax}, \text{lmic}_1)$ |
| (13) $(\binom{n}{2} - \text{sor}, -, \text{cyc}, \text{lmic}_1)$ | (14) $(\binom{n}{2} - \text{sor}, \text{lmic}_1, \text{lmax}, -)$ |
| (15) $(-, \text{cyc}, -, \text{rmin})$ | (16) $(-, \text{rmin}, -, \text{cyc})$ |
| (17) $(\text{maj}, \text{rmin}, \text{rmax}, -)$ | (18) $(\text{imaj}, \text{lmax}, \text{rmax}, -)$ |
| (19) $(\text{rmaj}, \text{lmax}, \text{lmin}, -)$ | (20) $(\text{chg}, \text{rmin}, \text{lmin}, -)$ |
| (21) $(\text{cochg}, \text{lmin}, \text{rmin}, -)$ | |

and the generating function with respect to the first three statistics in each quadruple is

$$F_n(q, x, y) = xy \prod_{r=2}^n (x + [r]_q + yq^{r-1} - 1 - q^{r-1}).$$

Proof. Observe that $\text{inv}(\sigma^{-1}) = \text{inv}(\sigma)$ and $\text{inv}(\sigma^r) = \text{inv}(\sigma^c) = \binom{n}{2} - \text{inv}(\sigma)$ for $\sigma \in \mathfrak{S}_n$. The proof is done via the following mappings.

$$\begin{aligned} (1) \xrightarrow{\phi} (2), \quad (1) \xrightarrow{i} (3), \quad (1) \xrightarrow{ioroc} (4), \quad (1) \xrightarrow{roc} (5), \quad (1) \xrightarrow{r} (6), \\ (1) \xrightarrow{c} (7), \quad (1) \xrightarrow{ior} (8), \quad (1) \xrightarrow{ioc} (9), \quad (3) \xrightarrow{\phi} (10), \quad (6) \xrightarrow{\phi} (11), \\ (6) \xrightarrow{\phi} (12), \quad (8) \xrightarrow{\phi} (13), \quad (9) \xrightarrow{\phi} (14), \quad (2) \xrightarrow{i} (15), \quad (10) \xrightarrow{i} (16) \\ (4) \xrightarrow{\psi} (17), \quad (17) \xrightarrow{i} (18), \quad (17) \xrightarrow{roc} (19), \quad (19) \xrightarrow{i} (20), \quad (20) \xrightarrow{r} (21). \end{aligned}$$

□

From the list we can read off many pairs of symmetric and joint equidistributed statistics.

Corollary 5.3. *In each of the following items, the pairs of statistics are symmetric and joint equidistributed over \mathfrak{S}_n .*

(1) $(\text{rmax}, \text{rmin})$, $(\text{rmax}, \text{lmax})$, $(\text{rmin}, \text{lmin})$, $(\text{lmax}, \text{lmin})$, $(\text{cyc}, \text{lmic}_1)$, and $(\text{lmic}_1, \text{lmax})$.

The generating function with respect to each pair is $F(1, x, y)$.

(2) $(\text{rmax}, \text{lmin})$, $(\text{rmin}, \text{lmax})$, $(\text{cyc}, \text{lmax})$, and $(\text{cyc}, \text{rmin})$.

6. CONCLUDING REMARKS

In this short paper we generalize simultaneously Petersen and Foata-Han's results to more than two statistics and find many triples or quadruples of statistics having the same joint distribution over \mathfrak{S}_n . In all quadruples, the first statistic is Mahonian while the others are Stirling, and we then read off many symmetric equidistributed pairs of Stirling statistics. However, it is well known that there are pairs of Mahonian statistics with a symmetric joint distribution as well [9], for examples, (inv, maj) and $(\text{maj}, \text{imaj})$ are two of them. Hence it would be interesting to generalize our results further to include more Mahonian statistics.

On the other hand, the generating function obtained in Theorem 1.3 only involves three of the four statistics, hence a natural question is to find a four-variable generating function including lmax as well. We leave them to the interested reader.

REFERENCES

- [1] A. Bjorner, F. Brenti, *Combinatorics of Coxeter Groups*, Springer, New York (2005)
- [2] W. Chen, G. Gong, and J. Guo, The sorting index and permutation codes, *Advances in Applied Mathematics* (2012).
- [3] R. Cori, Indecomposable permutations, hypermaps and labeled Dyck paths, *J. Combin. Theory Ser. A*, 116 (2009), pp. 1326-1343.
- [4] D. Foata, On the Netto inversion number of a sequence, *Proc. Amer. Math. Soc.* 19 (1968), pp. 236-240.
- [5] D. Foata, G.-H. Han, Signed words and permutations, I: a fundamental transformation. *Proc. Amer. Math. Soc.*, 135 (2007), pp. 31-40.
- [6] D. Foata, G.-H. Han, New permutation coding and equidistribution of set-valued statistics, *Theoret. Comput. Sci.*, 410 (2009), pp. 3743-3750.

- [7] D. H. Lehmer, Teaching combinatorial tricks to a computer, in: Proc. Sympos. Appl. Math., vol. 10, Amer. Math. Soc, Providence, RI, (1960), pp. 179-193.
- [8] Petersen, T. Kyle, The sorting index, Advances in Applied Mathematics 47.3 (2011): 615-630.
- [9] R.P. Stanley, Enumerative combinatorics. Vol. 49. Cambridge university press, 2011.
- [10] M.C. Wilson, An interesting new Mahonian permutation statistic, Electron. J. Combin., 17 (2010), p. R147.

DEPARTMENT OF MATHEMATICS, NATIONAL TAIWAN NORMAL UNIVERSITY, TAIPEI 11677, TAIWAN, ROC

E-mail address, Sen-Peng Eu: `speu@ntnu.edu.tw`

E-mail address, Yuan-Hsun Lo: `yhlo0830@gmail.com`

DEPARTMENT OF APPLIED MATHEMATICS, NATIONAL SUN YAT-SEN UNIVERSITY, KAOHSIUNG 80424, TAIWAN, ROC

E-mail address, Tsai-Lien Wong: `tlwong@math.nsysu.edu.tw`